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EPICENTER MOTION OF AN ELASTIC HALF-SPACE DUE TO BURIED STATIONARY AND MOVING LINE SOURCES*

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Abstract—A line source at $(0, y_0)$ is buried in an elastic half-space $(y > 0)$ whose surface is free of stress. The source points in an arbitrary direction and excites both P and S waves within the solid. Exact expressions are found for the horizontal and vertical displacements along the y-axis. Wave front singularities are identified and curves are presented for the epicenter displacements. Next, the buried line source, starting at a depth h, is allowed to move downward with constant speed c. By using the dynamic reciprocal theorem, the epicenter displacements for the moving source problem are found in terms of an integral. Explicit expressions are then given for the epicenter velocities which are displayed graphically.

1. INTRODUCTION

Two problems concerned with the propagation of waves in an elastic half-space are investigated in this paper. In the first an impulsive line body force is buried beneath the stress-free surface of the half-space. This source, which lies parallel to the free surface, has a δ-function time dependence and is of uniform strength along its length. The source acts in an arbitrary direction. Since the source does not vary along its axis, the elasticity problem, which is treated within the framework of linear elasticity, reduces to one of plane strain in the spatial coordinates x and y. By employing integral transforms, the author [1] has previously derived integral expressions for the horizontal and vertical components of displacement. Evaluation of these integrals by exact analytical means at an arbitrary interior point of the half-space is complicated by the wave conversion and reflection at the surface. The ingenious methods of Cagniard [2] and Pekeris [3] can, however, be used to easily investigate the body motion on the surface (considered in [1]) and along the line normal to the surface and passing through the buried source. These interior displacements along the y-axis are investigated in Section 2. There closed form expressions are given for the displacements and it is shown that the horizontal displacement is singular (i.e. becomes unbounded) at the S and SS wave fronts, while the vertical displacement is singular at the P and PP wave fronts. These infinite displacements, which violate the physical assumptions of linear elasticity (but not the mathematical equations of linear elasticity), are caused by the impulsive nature of the source. In a real physical situation the displacements would be (relatively) large but not infinite at the various singular wave fronts. Finally numerical results are presented for the time history of the epicenter displacements (on the surface directly above the buried line source) in Figs. 2 and 3.

Next, the response of an elastic half-space to a moving line source is studied in Section 3. The line starts at a depth h below the stress free surface and thereafter moves with a constant

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speed c along the y-axis and away from the surface (downward). The results of Section 2 are used as a Green's function, thus enabling the epicenter displacements to be expressed in terms of a single integral. Figures 4 and 5 give the time history of the epicenter velocities and also show the parametric dependence of these results on the moving load speed. Generally the velocity and displacement decrease with increasing c. It is also found that the epicenter displacements (but not velocities) remain bounded for all time for the moving load source considered here.

The motion of an elastic half-space, excited by a buried line source, has been studied previously by several authors. Garvin [4] treated the case where the source gave rise to P waves only and presented detailed numerical results for the surface response. Lapwood [5] used approximate techniques to study the disturbance when the depths of source and point reception were small compared with their distance apart. The surface motion of an elastic half-space due to an internally moving source has received considerable attention lately, especially by geophysicists, since this problem provides a model for earthquake generated waves caused by a moving fault. Ben-Menahem [6, 7] has treated several moving source problems, which incorporate a finite source and also curvature effects of the earth. A more detailed set of references for these problems (and more general elastic wave problems) can be found in the recent survey article by Miklowitz [8].

2. DISPLACEMENTS ON THE y-AXIS CAUSED BY A BURIED LINE SOURCE AT $(0, y_0)$

The region $-\infty < x, z < +\infty, y > 0$ is occupied by a homogeneous, isotropic, linear, elastic solid whose displacement vector $\mathbf{u}(x, y, z, t)$ is governed by the Navier equations

$$c_2^2\nabla^2\mathbf{u}+(c_1^2-c_2^2)\nabla(\nabla\cdot\mathbf{u}) = \mathbf{u}_{tt}-\frac{1}{\rho}\mathbf{F}. \tag{1}$$

$\mathbf{F}$ is the body force per unit volume and c_1 and c_2 are the speeds of propagation of dilatational (P wave) and equivoluminal (S wave) waves with $c_1 > c_2$. If the loading does not vary in the z direction the problem reduces to one of plane strain in which

$$\mathbf{u}(x, y, z, t) = \hat{x}u(x, y, t)+\hat{y}v(x, y, t). \tag{2}$$

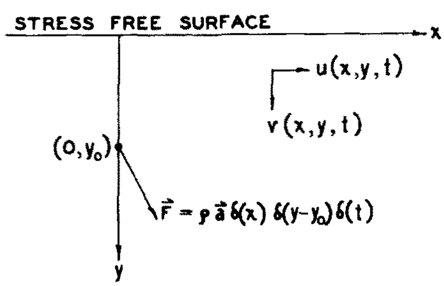

FIG. 1(a). Elastic half-space loaded by a buried line source $\mathbf{F}$.

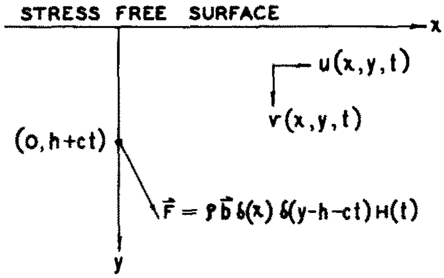

FIG. 1(b). Elastic half-space loaded by a downward moving line source **F**.

Suppose now that the elastic half-space, with stress free surface $y = 0$, is excited by the buried impulsive line force $\mathbf{F} = \rho\mathbf{a}\delta(x-x_0)\delta(y-y_0)\delta(t)$ (see Fig. 1a). The Navier equations (1) then reduce to the two scalar equations

$$c_2^2\nabla^2 u+(c_1^2-c_2^2)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = \frac{\partial^2 u}{\partial t^2}-a_1\delta(x-x_0)\delta(y-y_0)\delta(t) \tag{3}$$

and

$$c_2^2\nabla^2 v+(c_1^2-c_2^2)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial t^2}-a_2\delta(x-x_0)\delta(y-y_0)\delta(t) \tag{4}$$

where ∇^2 is the two dimensional Laplace operator and a_1 and a_2 are the x and y components of the vector **a**. The stress free boundary conditions

$$\sigma_{xy} = \sigma_{yy} = 0 \qquad \text{on } y = 0, \tag{5}$$

together with the initial conditions

$$u = \partial u/\partial t = v = \partial v/\partial t = 0 \qquad \text{at } t = 0, \tag{6}$$

when adjoined to the equations of motion (3) and (4) prescribe a well set problem.

In [1] the Fourier and Laplace transformed displacements were given as,

$$\begin{aligned} 2s^2\, e^{i\xi x_0}\bar{u}^* = {} & \left[-\frac{4mn}{m-n}A\, e^{-k_2y_0}-\frac{i\xi(m+n)}{k_1(m-n)}B\, e^{-k_1y_0}\right]e^{-k_1y}+[A\, e^{-k_2y_0}]\, e^{k_2y} \\ & +\left[\frac{i\xi}{k_1}B\, e^{-k_1y_0}\right]e^{k_1y}+\left[\frac{m+n}{m-n}A\, e^{-k_2y_0}+\frac{i\xi}{k_1(m-n)}B\, e^{-k_1y_0}\right]e^{-k_2y} \\ & \text{for } 0 \le y \le y_0. \end{aligned} \tag{7}$$

$$\begin{aligned} 2s^2\, e^{i\xi x_0}\bar{u}^* = {} & \left[-\frac{4mn}{m-n}A\, e^{-k_2y_0}-\frac{i\xi a}{k_1}B\, e^{k_1y_0}+(1-a)A\, e^{k_1y_0}-\frac{i\xi(m+n)}{k_1(m-n)}B\, e^{-k_1y_0}\right]e^{-k_1y} \\ & +\left[aA\, e^{k_2y_0}+\frac{i\xi(1+a)}{k_1}B\, e^{k_2y_0}+\frac{m+n}{m-n}A\, e^{-k_2y_0}+\frac{i\xi}{k_1(m-n)}B\, e^{-k_1y_0}\right]e^{-k_2y} \\ & \text{for } y_0 \le y < +\infty, \end{aligned} \tag{8}$$

$$2s^2\,\mathrm{e}^{i\xi x_0}\bar{v}^* = [B\,\mathrm{e}^{-k_1y_0}]\,\mathrm{e}^{k_1y}+\left[\frac{4mnk_1}{i\xi(m-n)}A\,\mathrm{e}^{-k_2y_0}+\frac{m+n}{m-n}B\,\mathrm{e}^{-k_1y_0}\right]\mathrm{e}^{-k_1y}$$
$$+\left[-\frac{i\xi}{k_2}A\,\mathrm{e}^{-k_2y_0}\right]\mathrm{e}^{k_2y}+\left[\frac{i\xi(m+n)}{k_2(m-n)}A\,\mathrm{e}^{-k_2y_0}-\frac{\xi^2}{k_1k_2(m-n)}B\,\mathrm{e}^{-k_1y_0}\right]\mathrm{e}^{-k_2y}$$
$$\text{for } 0\le y\le y_0, \tag{9}$$

and

$$2s^2\,\mathrm{e}^{i\xi x_0}\bar{v}^* = \left[\frac{4mnk_1}{i\xi(m-n)}A\,\mathrm{e}^{-k_2y_0}-\frac{i\xi(1+a)}{k_2}A\,\mathrm{e}^{k_1k_0}+\frac{m+n}{m-n}B\,\mathrm{e}^{-k_1y_0}\right.$$
$$\left.+aB\,\mathrm{e}^{k_1y_0}\right]\mathrm{e}^{-k_1y}+\left[\frac{i\xi a}{k_2}A\,\mathrm{e}^{k_2y_0}+(1-a)B\,\mathrm{e}^{k_2y_0}\right.$$
$$\left.+\frac{i\xi(m+n)}{k_2(m-n)}A\,\mathrm{e}^{-k_2y_0}-\frac{\xi^2}{k_1k_2(m-n)}B\,\mathrm{e}^{-k_1y_0}\right]\mathrm{e}^{-k_2y}\qquad\text{for } y_0\le y<+\infty, \tag{10}$$

where

$$m=\frac{k_2^2+\xi^2}{4k_1k_2},\qquad n=\frac{\xi^2}{k_2^2+\xi^2},\qquad a=\frac{k_1k_2+\xi^2}{k_1k_2-\xi^2},\qquad k_1=\sqrt{(\xi^2+s^2/c_1^2)},$$

$k_2=\sqrt{(\xi^2+s^2/c_2^2)}$, $A=a_1k_2-a_2i\xi$, and $B=a_1i\xi+a_2k_1$. The displacements u and v may be determined by means of the double inversion formula

$$u(x,y,t)=\frac{1}{4\pi^2 i}\int_\Gamma\int_{-\infty}^{+\infty}\bar{u}^*(\xi,y,s)\,\mathrm{e}^{st+i\xi x}\,\mathrm{d}\xi\,\mathrm{d}s,\ \text{etc.} \tag{11}$$

where the Laplace inversion path Γ, lies to the right of all singularities in the complex s-plane.

Attention will now be directed to the displacements on the line $x=x_0$, where (without loss of generality) x_0 is made to coincide with the origin $x=0$. The horizontal component of displacement, which from symmetry depends only on the horizontal component of the body force, is then given by

$$u(0,y,t)=\frac{a_1}{2\pi}\sum_{j=1}^{6}u^{(j)}(0,y,t). \tag{12}$$

The Laplace transform $\bar{u}^{(j)}(0,y,s)$ of the six separate parts of the horizontal displacement may be found from equations (7) and (8)

$$\bar{u}^{(1)}=-\int_0^\infty\frac{4mn}{s^2(m-n)}\mathrm{e}^{-k_2y_0-k_1y}\,\mathrm{d}\xi,\qquad \bar{u}^{(2)}=\int_0^\infty\frac{\xi^2(m+n)}{k_1s^2(m-n)}\mathrm{e}^{-(y+y_0)k_1}\,\mathrm{d}\xi,$$
$$\bar{u}^{(3)}=\int_0^\infty\frac{k_2}{s^2}\mathrm{e}^{-|y-y_0|k_2}\,\mathrm{d}\xi,\qquad \bar{u}^{(4)}=-\int_0^\infty\frac{\xi^2}{k_1s^2}\mathrm{e}^{-|y-y_0|k_1}\,\mathrm{d}\xi, \tag{13}$$
$$\bar{u}^{(5)}=\int_0^\infty\frac{k_2(m+n)}{s^2(m-n)}\mathrm{e}^{-(y_0+y)k_2}\,\mathrm{d}\xi,\quad\text{and}\quad \bar{u}^{(6)}=-\int_0^\infty\frac{\xi^2}{k_1s^2(m-n)}\mathrm{e}^{-k_1y_0-k_2y}\,\mathrm{d}\xi.$$

The expressions in equation (13) are valid throughout the full y range $y\ge 0$. The various waves $u^{(j)}$ which comprise the horizontal displacement component, may be identified from the exponential character of the integrands in (13). Using the terminology of seismologists [9]

$u^{(1)}$ is a SP wave, $u^{(2)}$ is a PP wave,

$u^{(3)}$ is a S wave, $u^{(4)}$ is a P wave,

$u^{(5)}$ is a SS wave, and $u^{(6)}$ is a PS wave.

In the absence of the stress free boundary $y = 0$ (i.e. for an unbounded elastic solid) only the pure S and P waves would be present. The other four waves arise by reflections of these S and P waves from the bounding surface.

The Laplace inversion of the various $\bar{u}^{(j)}$ can all be dealt with in a similar manner and only $\bar{u}^{(1)}$ will be considered here in detail, the others being of no greater difficulty. From equation (13)

$$\bar{u}^{(1)} = -\frac{1}{c_2^2}\int_0^\infty g(\eta)\,\mathrm{e}^{-(sy_0/c_2)f(\eta)}\mathrm{d}\eta, \tag{14}$$

where

$$g(\eta) = \frac{4\eta^2(2\eta^2+1)\sqrt{(\eta^2+1)}}{(2\eta^2+1)^2-4\eta^2\sqrt{(\eta^2+1)}\sqrt{(\eta^2+\alpha^2)}}, \qquad f(\eta) = \sqrt{(\eta^2+1)}+\beta\sqrt{(\eta+\alpha^2)},$$

$\alpha^2 = c_2^2/c_1^2, \beta = y/y_0$ and the integration variable ξ has been replaced by $s\eta/c_2$. The Laplace inversion of (14) is now immediate

$$u^{(1)} = -\frac{1}{c_2^2}\int_0^\infty g(\eta)\delta\left[t-\frac{y_0}{c_2}f(\eta)\right]\mathrm{d}\eta,$$

or

$$u^{(1)} = -\frac{1}{c_2 y_0}\int_0^\infty g(\eta)\delta[\theta-f(\eta)]\,\mathrm{d}\eta, \tag{15}$$

where $\theta = c_2 t/y_0$. It is easily established that there will be one root (say η_0) of $[\theta-f(\eta)]$ which lies within the integration range $(0, \infty)$, provided $\theta > (1+\alpha\beta)$, and no root otherwise. Here $(1+\alpha\beta)$ is just the minimum value of $f(\eta)$. Using well known properties of the Dirac δ-function, integration of (15) gives

$$u^{(1)} = -\frac{g(\eta_0)}{c_2 y_0 f'(\eta_0)}H[\theta-(1+\alpha\beta)] \tag{16}$$

where H is the Heaviside unit step function. The root η_0 is given by

$$\eta_0 = \left\{\frac{[\theta-\beta\sqrt{\{\theta^2-(1-\beta^2)(1-\alpha^2)\}}]^2}{(1-\beta^2)^2}-1\right\}^{1/2}$$

and furthermore

$$\sqrt{(\eta_0^2+1)} = \frac{\theta-\beta\sqrt{[\theta^2-(1-\beta^2)(1-\alpha^2)]}}{1-\beta^2}, \qquad \sqrt{(\eta_0^2+\alpha^2)} = \frac{\sqrt{[\theta^2-(1-\beta^2)(1-\alpha^2)]}-\theta\beta}{1-\beta^2},$$

and

$$f'(\eta_0) = \frac{\eta_0\sqrt{[\theta^2-(1-\beta^2)(1-\alpha^2)]}}{\sqrt{(\eta_0^2+1)}\sqrt{(\eta_0^2+\alpha^2)}},$$

so that

$$u^{(1)}(0, y, t) = -\frac{4\eta_0(2\eta_0^2+1)(\eta_0^2+1)\sqrt{(\eta_0^2+\alpha^2)}}{c_2 y_0\sqrt{[\theta^2-(1-\beta^2)(1-\alpha^2)]}\,[(2\eta_0^2+1)^2-4\eta_0^2\sqrt{(\eta_0^2+1)}\sqrt{(\eta_0^2+\alpha^2)}]} \times H[\theta-(1+\alpha\beta)]. \tag{17}$$

The remaining waves $u^{(j)}$ which make up $u(0, y, t)$ may be similarly inverted, so that only the final results will be quoted here.

Horizontal displacement $u(0, y, t)$ for $0 \le y < +\infty$.

$$u(0, y, t) = \frac{a_1}{2\pi}\sum_{j=1}^{6} u^{(j)} \quad \text{where}$$

$$u^{(1)} = -\frac{4\eta_0(2\eta_0^2+1)(\eta_0^2+1)\sqrt{(\eta_0^2+\alpha^2)}}{c_2\sqrt{[c_2^2t^2-(y_0^2-y^2)(1-\alpha^2)]}\,[(2\eta_0^2+1)^2-4\eta_0^2\sqrt{(\eta_0^2+1)}\sqrt{(\eta_0^2+\alpha^2)}]} \times H[c_2t-(y_0+\alpha y)]. \tag{18}$$

$$\eta_0 = \left\{\frac{[c_2ty_0-y\sqrt{\{c_2^2t^2-(y_0^2-y^2)(1-\alpha^2)\}}]^2}{(y_0^2-y^2)^2}-1\right\}^{\frac{1}{2}},$$

$$\left[=\left\{\frac{[c_2^2t^2+y_0^2(1-\alpha^2)]^2}{4c_2^2t^2y_0^2}-1\right\}^{\frac{1}{2}} \quad \text{when } y = y_0\right]$$

$$\sqrt{(\eta_0^2+1)} = \frac{c_2ty_0-y\sqrt{[c_2^2t^2-(y_0^2-y^2)(1-\alpha^2)]}}{y_0^2-y^2}, \qquad \left[=\frac{c_2^2t^2+y_0^2(1-\alpha^2)}{2c_2ty_0} \quad \text{when } y = y_0\right]$$

$$\sqrt{(\eta_0^2+\alpha^2)} = \frac{y_0\sqrt{[c_2^2t^2-(y_0^2-y^2)(1-\alpha^2)]}-c_2ty_0}{y_0^2-y^2}, \qquad \left[=\frac{c_2^2t^2-y_0^2(1-\alpha^2)}{2c_2ty_0} \quad \text{when } y = y_0\right]$$

$$u^{(2)} = \frac{\sqrt{[c_2^2t^2-\alpha^2(y+y_0)^2]}}{c_2(y+y_0)^2} \times \left[\frac{\{2c_2^2t^2-(2\alpha^2-1)(y+y_0)^2\}^2+4c_2t\{c_2^2t^2-\alpha^2(y+y_0)^2\}\sqrt{[c_2^2t^2+(1+\alpha^2)(y+y_0)^2]}}{\{2c_2^2t^2-(2\alpha^2-1)(y+y_0)^2\}^2-4c_2t\{c_2^2t^2-\alpha^2(y+y_0)^2\}\sqrt{[c_2^2t^2+(1-\alpha^2)(y+y_0)^2]}}\right] H[c_2t-\alpha(y_0+y)], \tag{19}$$

$$u^{(3)} = \frac{c_2^2t^2}{c_2(y-y_0)^2\sqrt{[c_2^2t^2-(y-y_0)^2]}}H[c_2t-|y-y_0|], \tag{20}$$

$$u^{(4)} = \frac{\sqrt{[c_2^2t^2-\alpha^2(y-y_0)^2]}}{c_2(y-y_0)^2}H[c_2t-\alpha|y-y_0|], \tag{21}$$

$$u^{(5)} = \frac{c_2^2t^2}{c_2(y+y_0)^2\sqrt{[c_2^2t^2-(y+y_0)^2]}}$$
$$\times\left[\frac{\{2c_2^2t^2-(y+y_0)^2\}^2+4c_2t\{c_2^2t^2-(y+y_0)^2\}\sqrt{[c_2^2t^2-(1-\alpha^2)(y+y_0)^2]}}{\{2c_2^2t^2-(y+y_0)^2\}^2-4c_2t\{c_2^2t^2-(y+y_0)^2\}\sqrt{[c_2^2t^2-(1-\alpha^2)(y+y_0)^2]}}\right]$$
$$\times H[c_2t-(y+y_0)], \qquad (22)$$

and

$$u^{(6)} = \frac{1}{c_2\sqrt{[c_2^2t^2+(y_0^2-y^2)(1-\alpha^2)]}}\left[\frac{4\eta_1(2\eta_1^2+1)(\eta_1^2+1)\sqrt{(\eta_1^2+\alpha^2)}}{(2\eta_1^2+1)^2-4\eta_1^2\sqrt{(\eta_1^2+\alpha^2)}\sqrt{(\eta_1^2+1)}}\right]$$
$$\times H[c_2t-(\alpha y_0+y)], \qquad (23)$$

$$\eta_1 = \left\{\frac{[c_2ty_0-y\sqrt{\{c_2^2t^2+(y_0^2-y^2)(1-\alpha^2)\}}]^2}{(y_0^2-y^2)^2}-\alpha^2\right\}^{\frac{1}{2}},$$

$$\left[=\left\{\frac{[c_2^2t^2-y_0^2(1-\alpha^2)]^2}{4c^2t^2y_0^2}-\alpha^2\right\}^{\frac{1}{2}} \quad \text{when } y=y_0\right]$$

$$\sqrt{(\eta_1^2+\alpha^2)} = \frac{c_2ty_0-y\sqrt{[c_2^2t^2+(y_0^2-y^2)(1-\alpha^2)]}}{y_0^2-y^2}, \quad \left[=\frac{c_2^2t^2-y_0^2(1-\alpha^2)}{2c_2ty_0} \quad \text{when } y=y_0\right]$$

$$\sqrt{(\eta_1^2+1)} = \frac{y_0\sqrt{[c_2^2t^2+(y_0^2-y^2)(1-\alpha^2)]}-c_2ty}{y_0^2-y^2}, \quad \left[=\frac{c_2^2t^2+y_0^2(1-\alpha^2)}{2c_2ty_0} \quad \text{when } y=y_0\right].$$

The singularities of $u(0, y, t)$ are identifiable from an inspection of equations (18)–(23). From the theory of linear hyperbolic partial differential equations [10] it is known that propagating singularities will occur only at wave fronts, so that the various wave types $u^{(j)}$ need only be examined at their respective wave fronts. $u^{(1)}$, $u^{(2)}$, $u^{(4)}$, and $u^{(6)}$ vanish at their wave fronts while $u^{(3)}$ and $u^{(5)}$ become unbounded, having reciprocal square root singularities at the S and SS wave fronts. Thus at any point on the y-axis two large horizontal displacements are experienced, the first at time $t = |y-y_0|/c_2$ due to the S front and later at time $t = (y+y_0)/c_2$ due to the reflected SS front.

By combining equations (18)–(23) and taking $y = 0$, the horizontal component of displacement is obtained at the epicenter ($x = y = 0$).

$$u(0,0,t) = \frac{a_1}{2\pi c_2y_0}[F_1(\theta)H(\theta-\alpha)+F_2(\theta)H(\theta-1)], \qquad (24)$$

where

$$F_1(\theta) = \frac{4\theta\sqrt{(\theta^2-\alpha^2)}\sqrt{[\theta^2+(1-\alpha^2)]}}{(2\theta^2+1-2\alpha^2)^2-4\theta(\theta^2-\alpha^2)\sqrt{(\theta^2+1-\alpha^2)}}, \qquad (25)$$

and

$$F_2(\theta) = \frac{2\theta^2(2\theta^2-1)}{\sqrt{(\theta^2-1)}[(2\theta^2-1)^2-4\theta(\theta^2-1)\sqrt{\{\theta^2-(1-\alpha^2)\}}]}, \qquad (26)$$

with $\theta = c_2t/y_0$. Figure 2 shows a plot of the displacement $u(0, 0, t)$ as a function of time. The initial horizontal displacement (after the P wave arrival) is to the left until $t = y_0/c_2$

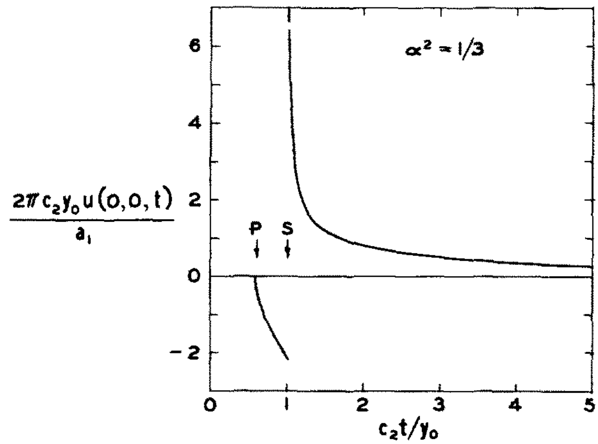

FIG. 2. Horizontal surface displacement vs. c_2t/y_0.

(the S wave arrival time) when the displacement suddenly jerks very far to the right and thereafter relaxes smoothly back to zero.

A large time estimate of the epicenter horizontal displacement may be obtained from equations (24)–(26), and is given by

$$u(0,0,t) = \frac{a_1}{2\pi c_2 y_0}\left[\frac{1}{(1-\alpha^2)\theta} + O(\theta^{-3})\right]. \tag{27}$$

This interesting result shows that $u(0,0,t)$ is, to the first order, independent of the depth y_0 of the buried line source for sufficiently large time. It might also be noted that the first order, large time approximation of the epicenter displacement is exactly the displacement for the half-plane problem with applied traction $\sigma_{xy} = (a_1/\rho)\delta(x)\delta(y)$ on $y = 0$†.

The above calculations for the horizontal displacement could now be repeated for the vertical displacement. Since these calculations involve nothing new, only the end results will be presented here.

Vertical displacement $v(0,0,t)$ for $0 \leq y < +\infty$

$$v(0,0,t) = \frac{a_2}{2\pi}\sum_{j=1}^{6} v^{(j)} \tag{28}$$

where

$$v^{(1)} = -\frac{4\eta_0(2\eta_0^2+1)(\eta_0^2+\alpha^2)\sqrt{(\eta_0^2+1)}}{c_2\sqrt{[c_2^2t^2-(y_0^2-y^2)(1-\alpha^2)]}\,[(2\eta_0^2+1)^2-4\eta_0^2\sqrt{(\eta_0^2+1)}\sqrt{(\eta_0^2+\alpha^2)}]} \times H[c_2t-(y_0+\alpha y)], \tag{29}$$

$$v^{(2)} = \frac{c_2^2t^2}{c_2(y_0+y)^2\sqrt{[c_2^2t^2-\alpha^2(y+y)^2]}} \times \left[\frac{\{2c_2^2t^2+(1-2\alpha^2)(y+y_0)^2\}^2+4c_2t\{c_2^2t^2-\alpha^2(y+y_0)^2\}\sqrt{[c_2^2t^2+(1-\alpha^2)(y+y_0)^2]}}{\{2c_2^2t^2+(1-2\alpha^2)(y+y_0)^2\}^2-4c_2t\{c_2^2t^2-\alpha^2(y+y_0)^2\}\sqrt{[c_2^2t^2+(1-\alpha^2)(y+y_0)^2]}}\right] \times H[c_2t-\alpha(y+y_0)], \tag{30}$$

† The author wishes to thank one of the reviewers for bringing this fact to his attention.

$$v^{(3)} = -\frac{\sqrt{[c_2^2t^2-(y_0-y)^2]}}{c_2(y_0-y)^2}H[c_2t-|y-y_0|], \tag{31}$$

$$v^{(4)} = \frac{c_2^2t^2}{c_2(y_0-y)^2\sqrt{[c_2^2t^2-\alpha^2(y_0-y)^2]}}H[c_2t-\alpha|y-y_0|], \tag{32}$$

$$v^{(5)} = \frac{\sqrt{[c_2^2t^2-(y_0+y)^2]}}{c_2(y+y_0)^2} \times \left[\frac{\{2c_2^2t^2-(y+y_0)^2\}^2+4c_2t\{c_2^2t^2-(y+y_0)^2\}\sqrt{[c_2^2t^2-(1-\alpha^2)(y+y_0)^2]}}{\{2c_2^2t^2-(y+y_0)^2\}^2-4c_2t\{c_2^2t^2-(y+y_0)^2\}\sqrt{[c_2^2t^2-(1-\alpha^2)(y+y_0)^2]}}\right] \times H[c_2t-(y+y_0)], \tag{33}$$

and

$$v^{(6)} = \frac{1}{c_2\sqrt{[c_2^2t^2+(y_0^2-y^2)(1-\alpha^2)]}}\left[\frac{4\eta_1(2\eta_1^2+1)(\eta_1^2+\alpha^2)\sqrt{(\eta_1^2+1)}}{(2\eta_1^2+1)^2-4\eta_1^2\sqrt{(\eta_1^2+\alpha^2)}\sqrt{(\eta_1^2+1)}}\right] \times H[c_2t-(\alpha y_0+y)]. \tag{34}$$

In the above η_0 and η_1 have the same definitions as previously given in equations (18) and (23). Also the number superscript on $v^{(j)}$ denotes the same wave type as that of the corresponding $u^{(j)}$, e.g. $v^{(1)}$ is an *SP* wave. The $v^{(2)}$ and $v^{(4)}$ waves carry reciprocal square root singularities at their wave fronts, while the remaining $v^{(j)}$ vanish at their fronts. Thus the vertical displacement at any point on the y-axis will be large at two distinct times, the first at time $t = |y-y_0|/c_1$ due to the *P* front and later at time $t = (y+y_0)/c_1$ due to the reflected *PP* front.

Equations (28)–(34) may be combined to give the vertical component of displacement at the epicenter

$$v(0,0,t) = \frac{a_2}{2\pi c_2 y_0}[G_1(\theta)H(\theta-\alpha)+G_2(\theta)H(\theta-1)], \tag{35}$$

where

$$G_1(\theta) = \frac{2\theta^2(2\theta^2+1-2\alpha^2)}{\sqrt{(\theta^2-\alpha^2)}[(2\theta^2+1-2\alpha^2)^2-4\theta(\theta^2-\alpha^2)\sqrt{\{\theta^2+(1-\alpha^2)\}}]}, \tag{36}$$

and

$$G_2(\theta) = \frac{4\theta\sqrt{(\theta^2-1)}\sqrt{[\theta^2-(1-\alpha^2)]}}{(2\theta^2-1)^2-4\theta(\theta^2-1)\sqrt{[\theta^2-(1-\alpha^2)]}}. \tag{37}$$

Figure 3 shows a plot of the displacement $v(0, 0, t)$ as a function of time. Upon arrival of the initial *P* wave the vertical displacement suddenly jumps downward and then quickly rises to a local maximum before again falling to a minimum for the *S* wave arrival. The curve exhibits a kink at the *S* wave front since the velocity $\dot{v}(0, 0, t)$ has a jump there. After the *S* wave has passed, the vertical displacement gradually relaxes back to zero. Unlike the horizontal displacement, the vertical displacement is always of the same sign (i.e. positive or downward).

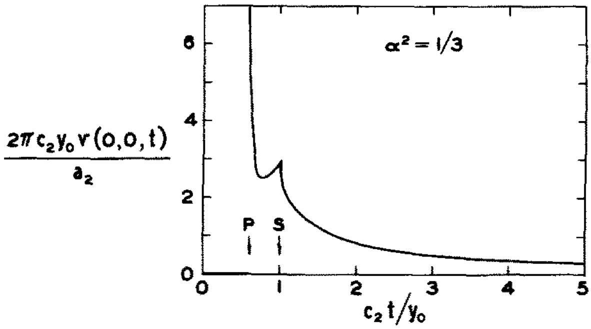

FIG. 3. Vertical surface displacement vs. c_2t/y_0.

For large time, the epicenter vertical displacement behaves as,

$$v(0,0,t) = \frac{a_2}{2\pi c_2 y_0}\left[\frac{1}{(1-\alpha^2)\theta} + O(\theta^{-3})\right], \tag{38}$$

so that $v(0,0,t)$ also tends to zero as t^{-1} and is independent of the depth of the buried source, for sufficiently large time. Note also that $u(0,0,t)/a_1 = v(0,0,t)/a_2$ to the first order as $t \to +\infty$.

3. SURFACE VELOCITIES DIRECTLY ABOVE A DOWNWARD MOVING LINE-SOURCE

If the buried source considered in Section 2 is now allowed to move downward (see Fig. 1b), the corresponding elastic wave problem is more complicated due to the moving excitation. As with a previous problem [1] for the surface response due to a moving line load, integral expressions for the displacements are most quickly obtained by the use of the dynamic reciprocal theorem. This theorem equates the work done by the body forces $\mathbf{F}$ acting through displacements $\mathbf{u}'$ to that of the body forces $\mathbf{F}'$ acting through displacement $\mathbf{u}$. Thus

$$\int_0^t \int_0^\infty \int_{-\infty}^\infty \mathbf{F}(\mathbf{x},\tau)\cdot\mathbf{u}(\mathbf{x},t-\tau)\,dx\,dy\,d\tau = \int_0^t \int_0^\infty \int_{-\infty}^\infty \mathbf{F}(\mathbf{x},\tau)\cdot\mathbf{u}(\mathbf{x},t-\tau)\,dx\,dy\,d\tau. \tag{39}$$

The unprimed and primed problems are specified by

Unprimed problem

$$c_2^2\nabla^2\mathbf{u} + (c_1^2 - c_2^2)\nabla(\nabla\cdot\mathbf{u}) = \mathbf{u}_{tt} - \mathbf{b}\delta(x)\delta[y-\phi(t)]H(t) \text{ in } -\infty < x < +\infty,$$

$y > 0$, with stress free boundary conditions on $y = 0$, displacement vector $\mathbf{u} = \hat{x}u(x,y,t) + \hat{y}v(x,y,t)$, moving body force $\mathbf{F} = \rho\mathbf{b}\delta(x)\delta\,[y-\phi(t)]H(t)$ and $\mathbf{b} = \hat{x}b_1 + \hat{y}b_2$.

Primed problem

$$c_2^2\nabla^2\mathbf{u}' + (c_1^2 - c_2^2)\nabla(\nabla\cdot\mathbf{u}') = \mathbf{u}'_{tt} - \mathbf{a}\delta(x-x_0)\delta(y-y_0)\delta(t) \text{ in } -\infty < x < +\infty,$$

$y > 0$, with stress free boundary conditions on $y = 0$, displacement vector

$$\mathbf{u}' = \hat{x}u'(x, y; x_0, y_0, t) + \hat{y}v'(x, y; x_0, y_0, t),$$

stationary body force $\mathbf{F}' = \rho\mathbf{a}\delta(x-x_0)\delta(y-y_0)\delta(t)$ and $\mathbf{a} = \hat{x}a_1 + \hat{y}a_2$. Both of these problems have zero initial conditions.

Applying the reciprocal theorem gives

$$\int_0^t \int_0^\infty \int_{-\infty}^\infty \{\rho\mathbf{b}\delta(x)\delta[y-\phi(\tau)]H(\tau)\} \cdot \{\hat{x}u'(x, y; x_0, y_0, t-\tau) + \hat{y}v'(x, y; x_0, y_0, t-\tau)\}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\tau$$

$$= \int_0^t \int_0^\infty \int_{-\infty}^\infty \{\rho\mathbf{a}\delta(x-x_0)\delta(y-y_0)\delta(\tau)\} \cdot \{\hat{x}u(x, y, t-\tau) + \hat{y}v(x, y, t-\tau)\}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\tau$$

or

$$a_1u(x_0, y_0, t) + a_2v(x_0, y_0, t) = b_1 \int_0^t u'[0, \phi(\tau); x_0, y_0, t-\tau]\, \mathrm{d}\tau + b_2 \int_0^t v'[0, \phi(\tau); x_0, y_0, t-\tau]\, \mathrm{d}\tau. \quad (40)$$

It should be emphasized here that the primed displacements required for the integrand of (40) are more general than the displacements (12) and (28) found in Section 2. This is so because in previously taking $x = x_0 = 0$, the independence of the two parameters x and x_0 was lost. For this reason only the epicenter motion due to the moving load will be studied. In this case $x_0 = y_0 = 0$ so that

$$a_1u(0, 0, t) + a_2v(0, 0, t) = b_1 \int_0^t u'[0, \phi(\tau); 0, 0, t-\tau]\, \mathrm{d}\tau + b_2 \int_0^t v'[0, \phi(\tau); 0, 0, t-\tau]\, \mathrm{d}\tau.$$

Writing $u' = a_1u''$, $v' = a_2v''$ and realizing that a_1 and a_2 may vary independently of one another gives,

$$u(0, 0, t) = b_1 \int_0^t u''[0, \phi(\tau); 0, 0, t-\tau]\, \mathrm{d}\tau, \quad (41)$$

and

$$v(0, 0, t) = b_2 \int_0^t v''[0, \phi(\tau); 0, 0, t-\tau]\, \mathrm{d}\tau. \quad (42)$$

Here

$$u''[0, \phi(\tau); 0, 0, t-\tau] = \frac{1}{2\pi} \sum_{j=1}^{6} u^{(j)} \quad (43)$$

and

$$v''[0, \phi(\tau); 0, 0, t-\tau] = \frac{1}{2\pi} \sum_{j=1}^{6} v^{(j)} \quad (44)$$

where in the expressions for $u^{(j)}$ (equations (18)–(23)) and $v^{(j)}$ (equations (29)–(34)) y must be replaced by $\phi(\tau)$, y_0 by zero, and t by $t-\tau$.

When the moving source starts at a depth h at time $t = 0$ and thereafter moves downward with a constant speed c, the appropriate form for $\phi(\tau)$ is

$$\phi(\tau) = h+c\tau. \tag{45}$$

The substitution $\xi = c_2(t-\tau)/\phi(\tau)$ then allows the epicenter displacements to be written as,

$$\begin{aligned}
0 \le c_2t/h \le \alpha \qquad & u(0,0,t) = 0,\\
\alpha \le c_2t/h \le 1 \qquad & u(0,0,t) = \frac{b_1}{2\pi c_2^2}\int_{\alpha}^{c_2t/h}\frac{F_1(\xi)\,d\xi}{\omega\xi+1},\\
1 \le c_2t/h \qquad & u(0,0,t) = \frac{b_1}{2\pi c_2^2}\int_{\alpha}^{c_2t/h}\frac{F_1(\xi)\,d\xi}{\omega\xi+1}+\frac{b_1}{2\pi c_2^2}\int_{1}^{c_2t/h}\frac{F_2(\xi)\,d\xi}{\omega\xi+1},
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
0 \le c_2t/h \le \alpha \qquad & v(0,0,t) = 0,\\
\alpha \le c_2t/h \le 1 \qquad & v(0,0,t) = \frac{b_2}{2\pi c_2^2}\int_{\alpha}^{c_2t/h}\frac{G_1(\xi)\,d\xi}{\omega\xi+1},\\
1 \le c_2t/h \qquad & v(0,0,t) = \frac{b_2}{2\pi c_2^2}\int_{\alpha}^{c_2t/h}\frac{G_1(\xi)\,d\xi}{\omega\xi+1}+\frac{b_2}{2\pi c_2^2}\int_{1}^{c_2t/h}\frac{G_2(\xi)\,d\xi}{\omega\xi+1},
\end{aligned} \tag{47}$$

where $\omega = c/c_2$ is the (constant) ratio of the moving load speed to that of the S wave.

Equations (46) and (47) present, in integral form, expressions for the surface displacements directly above a buried line source which moves downwards along the y-axis with a constant speed c after starting from the initial depth h. Although it should be possible to perform these integrations in closed form (similar integrals having been considered in [1]), the calculations would be lengthy and for this reason only the epicenter velocities $\dot{u}(0,0,t)$ and $\dot{v}(0,0,t)$ will be discussed here.

By taking the time derivative of equations (46) and (47), the epicenter velocities are easily found to be

$$\dot{u}(0,0,\mathrm{t}) = \frac{b_1}{2\pi c_2 h}\left[\frac{F_1(\xi)}{\omega\xi+1}H(\xi-\alpha)+\frac{F_2(\xi)}{\omega\xi+1}H(\xi-1)\right], \tag{48}$$

and

$$\dot{v}(0,0,t) = \frac{b_2}{2\pi c_2 h}\left[\frac{G_1(\xi)}{\omega\xi+1}H(\xi-\alpha)+\frac{G_2(\xi)}{\omega\xi+1}H(\xi-1)\right], \tag{49}$$

where $\xi = c_2t/h$. Figures 4 and 5 show plots of $\dot{u}$ and $\dot{v}$ for $\alpha^2 = \frac{1}{3}$ and various values of the moving load speed c. These curves show that the epicenter velocities for the moving load problem have the same qualitative features as the epicenter displacements corresponding to the stationary load problem of Section 2. (Compare Figs. 4 and 5 with Figs. 2 and 3.) In fact, if the moving load speed is reduced to zero ($\omega = 0$) then the corresponding figures become identical. That this should be so follows from the step function time dependence of the moving load problem (when the moving load speed is zero) as opposed to the δ-function

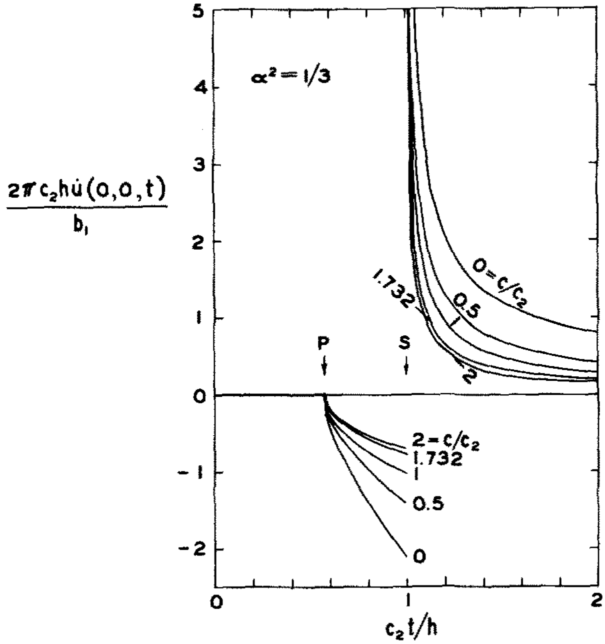

FIG. 4. Horizontal surface velocity vs. c_2t/h for various moving load speeds c.

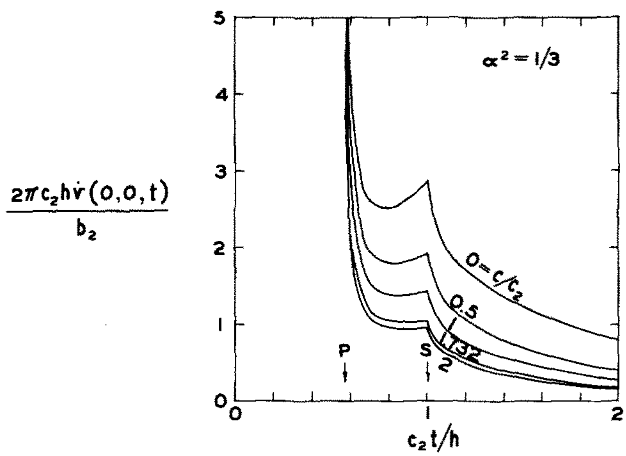

FIG. 5. Vertical surface velocity vs. c_2t/h for various moving load speeds c.

time dependence of the stationary impulse problem. The figures also indicate that an increase in the load speed c causes a decrease in the surface response above the load. Finally, it should be noted that the displacements obtained by integrating equations (48) and (49), (i.e. equations (46) and (47)) will not contain singularities (infinite displacements) since the singularities of the epicenter velocities are integrable.

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Абстракт—Линейный источник при /0, y_0/ находится в упругом полупространстве /$y > 0$/, поверхность которого свободна от напряжений. Источник направлен в произельном напъавленнии и излучает волны как P пак и S в твердом теле. Определяются точные вырахения для горизонталъных и вертикалъных перемещений вдолъ оси y. Определяются сингулярности фронта волны и приводятся кривые для перемещения эпицентра. Далее, этотже линейный источник, начиная с глубины h, может передвигатся вниз с постоянной скоростью c. С помощью обратной динамической теоремы определяются перемещения эпицентра для задачи с движущимся источником в виде некорого интеграла. Приводятся, затем, выражения, в явном виде, для скорости эпицентра, представленых графически.